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POLYANALYTIC FUNCTIONS ON BANACH SPACES

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Abstract

We generalize to infinite dimensions the concept of polyanalytic functions. These are our main results: (1) A characterization based upon restriction, which generalizes a known characterization of holomorphic functions on Banach spaces. (2) A property of special local uniform limits which yields an approximation result. (3) We introduce meta-analytic functions on Hilbert manifolds together with a characterization compatible with (1). We also point out several corollaries to our characterizations.

1. Introduction

There is a well-established field of research on infinite dimensional holomorphy, see, e.g., the books of Dineen [8] and Mujica [15]. Two basic examples of properties which have been investigated are the problem of approximation of a holomorphic function by entire functions (e.g., it is

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known that a holomorphic function on the unit ball centered at the origin in a complex Banach manifold can be approximated by entire functions on smaller balls centered at the origin, see Lempert [14]) and the Levi problem, see overview in Dineen [8], and we mention that some results concerning holomorphic extension of mappings from compacts are know, which have certain additional requirements, e.g., existing holomorphic extension to a neighbourhood of the compact for all compositions with dual elements of the target space, see Khue & Tac [11]. We shall be concerned with a generalization of q-analytic functions (see the survey article Balk [2] and the references therein).

Definition 1.1 (Polyanalytic function of order q, q-analytic function)**.** Let $U \subset \mathbb{C}$ be a domain. A function f on U is called *q-analytic* on U if it has locally near any $p \in U$, the representation

$$f(z) = \sum_{j=0}^{n-1} a_j(z;\ p)(\bar{z} - \bar{p})^j, \tag{1}$$

with holomorphic $a_j(z;\ p)$. A function f is called *countably analytic* on U if for every point $a \in U$, there is a neighbourhood U_p of p such that $f(z) = \sum_{k=0}^{\infty} h_k(z;\ p)(\bar{z} - \bar{p})^k$ for holomorphic $h_k(z;\ p)$ on U_p.

Results. We provide a generalization of q-analytic functions to Banach manifolds. For such functions, we prove a characterization of q-analytic functions on Banach manifolds (see Proposition 2.6) and a property of special local uniform limits (see Proposition 2.19). We also point out some consequences of the characterization. Then we introduce meta-analytic functions on Banach manifolds and give a characterization compatible to that of the q-analytic functions (see Corollary 3.7) based upon restriction. Also here we point out some consequences of the characterization.

We provide the basic notions of infinite dimensional holomorphy in Appendix A. In this text, we shall always assume Fréchet holomorphy, i.e., any function which is called holomorphic is assumed to be locally bounded in this text. Here, we shall begin by introduction of q-analytic functions on Banach manifolds.

2. Absolute q-Analytic Functions on Banach Manifolds

This article concerns q-analytic functions in an infinite-dimensional setting. For this we use monomials in conjugate variables.

Definition 2.1. Let X be a complex Banach space and let $\mathcal{P}({}^m X, \mathbb{C})$ denote the space of m-homogeneous polynomials in the sense of Definition A.2. Denote

$$\overline{\mathcal{P}}({}^m X, \mathbb{C}) = \{\psi : X \to \mathbb{C} \text{ such that } \psi = \overline{\phi(z)}, \text{ some } \phi \in \mathcal{P}({}^m X, \mathbb{C})\}. \quad (2)$$

Definition 2.2 (Functions of absolute order q)**.** Let X be a Banach manifold. A function $f \in C^{\infty}(X, \mathbb{C})$ is called *polyanalytic of absolute order* q or *absolute* q*-analytic* at the origin if in a neighbourhood U_0 of the origin in X, f has the representation

$$f(z) = \sum_{m=0}^{q-1} \sum_{\mathfrak{m}_m \in B_m} a_{m, \mathfrak{m}_m}(z)\mathfrak{m}_m, \; a_{m, \mathfrak{m}_m} \in \mathcal{O}(U_0), \; \mathfrak{m}_m \in B_m \subset \overline{\mathcal{P}}({}^m X, \mathbb{C}), \quad (3)$$

where B_m is a linearly independent basis for $\overline{\mathcal{P}}({}^m X, \mathbb{C})$, f is called *countably analytic* at 0 if it has the representation,

$$f(z) = \sum_{m=0}^{\infty} \sum_{\mathfrak{m}_m \in B_m} a_{m, \mathfrak{m}_m}(z)\mathfrak{m}_m, \quad (4)$$

and if the required local representation but with translation of the origin holds near every point we simply call f absolute q-analytic or countably analytic respectively.

2.1. A characterization based upon restriction

Obviously any absolute q-analytic function is an absolute $(q+1)$-analytic function. If X is a complex Banach space and $V \subset X$ a finite dimensional complex subvector space, then $P \in \overline{\mathcal{P}}({}^m X, \mathbb{C})$ implies that the restriction $P|_V \in \overline{\mathcal{P}}({}^m V, \mathbb{C})$. If we, for instance, denote by $z := (z_1, \dots, z_n)$ the complex variables in V, then any holomorphic a_β on X restricts to a function which is holomorphic in z on the finite dimensional complex Euclidean manifold canonically obtained from the complex vector space V.

Observation 2.3. By the homogeneity properties of any member $\mathfrak{m}_m \in \overline{\mathcal{P}}({}^m X, \mathbb{C})$ its restriction to V has the form of a sum of monomials $\bar{z}_1^{\alpha_1} \dots \bar{z}_n^{\alpha_n}$, $|\alpha| = \sum_j \alpha_j \leq m$, i.e., is a sum of elements in $\bigcup_{j=1}^m \overline{\mathcal{P}}({}^j V, \mathbb{C})$. Whence the restriction to V of an absolute q-analytic function (near the origin) has the form $\sum_{|\beta| \leq q-1} a_\beta(z)\bar{z}^\beta$ (for some holomorphic a_β near the origin).

Observation 2.4. Fixing the variables z_j, $j \neq k, 1 \leq j \leq n$, any function of the form $\sum_{|\beta| \leq q-1} a_\beta(z)\bar{z}^\beta$ reduces to a q-analytic function in the variable z_k. Because the restriction of any absolute q-analytic function on X restricts to an absolute $n \cdot q$-analytic function on any finite n-dimensional V and by Observation 2.3 we see that being absolute

q-analytic on X implies being q-analytic (in the one-dimensional sense of Definition 1.1) along each one complex dimensional slice.

Note that we intentionally use the term *absolute* order (instead of merely the term order) because of the absence of conditions specified to the separate components of the variable.

Example 2.5. Let $X = \mathbb{C}^2$ which we canonically identify as a Euclidean complex manifold and denote $z = (z_1, z_2) \in \mathbb{C}^2$ the complex coordinates. The two functions $f_1(z) = \bar{z}_1^4\bar{z}_2^3$ and $f_2(z) = \bar{z}_1^3\bar{z}_2^4$ both belong to $\overline{\mathcal{P}}({}^7X, \mathbb{C})$ and one can easily see that they are absolute 8-analytic. However $\left(\frac{\partial}{\partial\bar{z}_2}\right)^4 f_1 \equiv 0 \not\equiv \left(\frac{\partial}{\partial\bar{z}_2}\right)^4 f_2$. Thus if one does not use the term *absolute* order, then f_1 must be said to be of order $\alpha = (5, 4)$ whereas f_2 should be of order $\alpha = (4, 5)$.

Proposition 2.6. *Let X be a complex Banach space with countable basis* (*in particular a complex Banach manifold with open unit ball and a single chart*), *let* $U \subset X$ *be open and let* $f \in C^q(U, \mathbb{C})$. *Then f is absolute q-analytic on U iff the restriction of f to any one-dimensional complex slice is q-analytic in the sense of Definition* 1.1.

Proof. The "only if"-direction follows from Observation 2.4. So assume the restriction of f to any one dimensional complex slice is q-analytic. We now use induction in $q \in \mathbb{Z}_+$. For $q = 1$, the result is known since locally bounded functions (in the case $q = 1$ we can assume C^1-smooth) functions are holomorphic iff they are holomorphic along each complex line, see, e.g., Dineen [8], p. 144. Assume $q > 1$ and any C^{q-1}-smooth function which is $(q-1)$-analytic along every one-dimensional slice is absolute $(q-1)$-analytic on U. Let $f \in C^q(U)$ such that f is q-analytic along each one-dimensional complex slice. Assume w.l.o.g. $0 \in U$. Since X has a countable basis every element $z \in X$ shall be represented as $z = (z^{(i)})_{i \in \mathbb{N}}$ and we denote the unit sphere by $S = \{z \in X : \|z\| = 1\}$. Define

$$\mathscr{A} = \{w \in S : (w^{(i)} = 0,\, i = 1, \dots, n-1 \text{ and } w^{(n)} \neq 0) \Rightarrow (\,\mathrm{Im} w^{(n)} = 0)\}. \tag{5}$$

We can decompose

$$X = \bigcup_{w \in \mathscr{A}} \lambda_w,\ \lambda_w := \{\zeta \cdot w : \zeta \in \mathbb{C}\}. \tag{6}$$

This means that (because f is q-analytic along every line) there are functions $a_{i,w}(\zeta \cdot w)$ which are holomorphic in the variable ζ such that we can write

$$f(\zeta \cdot w) = \sum_{i=0}^{q-1} a_{i,w}(\zeta \cdot w)\bar{\zeta}^i,\ \zeta \in \mathbb{C},\ w \in \mathscr{A},\ \zeta \cdot w \in U. \tag{7}$$

Let f be defined according to Equation (7) such that f is q-analytic along every line in U, and assume (as our induction hypothesis) that any such function with q replaced by $(q-1)$ automatically defines an absolute $(q-1)$-analytic function. Recall that for any $z \in U$, there exists $\zeta_z \in \mathbb{C}$ and $w_z \in \mathscr{A}$ such that $z = \zeta_z \cdot w_z$ and consider the function $h''(z) = h''(\zeta_z w_z) := \overline{\zeta_z}$, $\forall z \in U$. Obviously $\overline{h''} \in \mathcal{P}(^1U, \mathbb{C})$ since,

$$\overline{h''(\zeta \cdot z)} = \overline{h''(\zeta \cdot \zeta_z w_z)} := \zeta \cdot \zeta_z = \zeta \cdot \overline{h''(z)}. \tag{8}$$

Letting

$$h(\zeta \cdot w) := a_{0,w}(\zeta \cdot w),\ w \in \mathscr{A},\ \zeta \in \mathbb{C},\ \zeta \cdot w \in U, \tag{9}$$

we can decompose

$$f(z) = h(z) + h'(z) \cdot h''(z), \quad z \in U. \tag{10}$$

By definition $h(\zeta \cdot w) = a_{0,w}(\zeta \cdot w)$ is already holomorphic along each λ_w. We first show that h defines a holomorphic function, which implies that $f - h = h' \cdot h''$ is q-analytic along each complex line in U. Then necessarily (because h'' belongs to $\overline{\mathcal{P}}(^1U, \mathbb{C})$, i.e., homogeneous anti-linear of degree one) $h'(z)$ defines a function which is $(q-1)$-analytic along every line in U thus satisfies the conditions of the induction hypothesis, i.e., defines an absolute $(q-1)$-analytic function. To see that h is holomorphic let λ be a complex one-dimensional slice in U. For any two points $w_1, w_2 \in \mathscr{A}$, the space $M := \mathrm{Span}_{\mathbb{C}}\{w_1, w_2\}$ is a complex two dimensional vector space and Euclidean manifold in which $\{w_1, w_2\}$ form an orthonormal Euclidean basis. Denote by (ζ_1, ζ_2) holomorphic coordinates (with respect to the orthonormal basis $\{w_1, w_2\}$) for M centered at the origin. Since f is q-analytic along each one dimensional complex slice of U (by assumption) we must have that $f|_M$ is separately holomorphic with respect to (ζ_1, ζ_2) in $U \cap M$.

The following is a known generalization of Hartogs' theorem in finite dimension.

Theorem 2.7 (Avanissian & Traoré [1], Theorem 1.3, p. 264)**.** *Let* $\Omega \subset \mathbb{C}^n$ *be a domain and let* $z = (z_1, \dots, z_n)$, *denote holomorphic coordinates in* $\mathbb{C}^n$ *with* $\mathrm{Re}\, z =: x, \mathrm{Im}\, z = y$. *Let f be a function which, for each j, is smooth in* x_j, y_j *and polyanalytic of order* α_j *in the variable* $z_j = x_j + iy_j$ *(in such case we shall simply say that f is separately* *polyanalytic of order* α). *Then f is jointly smooth with respect to* (x, y) *on* Ω *and furthermore is polyanalytic of order* $\alpha = (\alpha_1, \dots, \alpha_n)$.

By Theorem 2.7, $f|_M$ is jointly q-analytic w.r.t. (ζ_1, ζ_2) that is,

$$f|_M(\zeta_1 \cdot w_1 + \zeta_2 \cdot w_2) =: f|_M(\zeta_1, \zeta_2) = \sum_{0 \le i+k < q} a_{i,k}(\zeta_1, \zeta_2)\overline{\zeta}_1^i \overline{\zeta}_2^k, \tag{11}$$

where $a_{i,k}(\zeta_1, \zeta_2)$ is holomorphic on $U \cap M$ and $a_{i,k}(\zeta_1, 0) = a_{i,w_1}(\zeta_1 \cdot w_1)$, $a_{i,k}(0, \zeta_2) = a_{i,w_2}(\zeta_2 \cdot w_2)$. Let w_1, w_2 be chosen such that $\lambda \subset M \cap U$. Now we can cover $M \cap U$ by a union of lines λ_w, $w \in \mathscr{A} \cap M$ and w.l.o.g. there exists $c, d \in \mathbb{C}$ such that,

$$\lambda_w = \{\zeta_1 \cdot d \cdot w = \zeta_1 \cdot (w_1 + c \cdot w_2),\ \zeta_1 \in \mathbb{C}\}. \tag{12}$$

Now on the one hand,

$$f(\zeta_1 \cdot d \cdot w) = a_{0,w}(\zeta_1 \cdot d \cdot w) + \sum_{i=1}^{q-1} a_{i,w} \cdot \overline{d}^i \cdot \overline{\zeta}_1^i, \tag{13}$$

and on the other hand (since $\lambda_w \in M$), Equation (11) gives

$$f(\zeta_1 \cdot d \cdot w) = f(\zeta_1 \cdot (w_1 + c \cdot w_2)) = a_{0,0}(\zeta_1 \cdot d \cdot w) + \sum_{1 \le i+k < q} a_{i,k}(\zeta_1, c \cdot \zeta_1)\overline{c}^k \cdot \overline{\zeta}_1^{i+k}. \tag{14}$$

This implies that

$$a_{0,0}|_{\lambda_w}(z) = a_{0,w}|_{\lambda_w}(z) = h(z), \quad \forall z \in \lambda_w,\ w \in \mathscr{A} \cap \subset M, \tag{15}$$

hence (because the λ_w, $w \in \mathscr{A} \cap \subset M$ cover $M \cap U$) h is holomorphic along $\lambda \subset M$ in U, since $a_{0,0}$ is holomorphic on $M \cap U$. Since λ was an arbitrary one-dimensional complex slice in U this yields that h is a holomorphic function on U. Finally, the function $f = h + h' \cdot h''$ is the sum of a holomorphic function and an absolute q-analytic function on U thus itself an absolute q-analytic function on U. This completes the proof.

$\square$

A consequence of Proposition 2.6 is the following result on zero sets of absolute q-analytic functions.

Proposition 2.8. *Let X be a complex Banach space with countable basis and let f be a $C^{2q-1}(U, \mathbb{C})$-smooth function on an open neighbourhood U of 0 in X which is absolute q-analytic on $U \setminus f^{-1}(0)$.*

Proof. Let λ be an arbitrary one-dimensional complex slice in U (by which we mean that λ is the intersection with U of the complex span in X of some vector in X). If we assume $f \in C^{2q-1}(U, \mathbb{C})$ such that f is absolute q-analytic on $U \setminus f^{-1}(0)$, then we know that $f|_{\lambda}$ is a q-analytic function on $\lambda \cap (U \setminus f^{-1}(0))$.

The following two theorems (it was proved for holomorphic functions in one variable by Radó [16] and generalized to several variables by Cartan [5]) are known for polyanalytic functions of several variables.

Theorem 2.9 (Daghighi & Wikström [7]). *Let $\Omega \subset \mathbb{C}^n$ be a bounded, simply connected domain. Let $\alpha \in \mathbb{Z}_+^n$ and let f be a function on Ω which is separately $C^{2\alpha_j-1}$-smooth with respect to the z_j-variable. If f is α-analytic on $\Omega \setminus f^{-1}(0)$, then f is α-analytic on Ω.*

If instead $f \in C^{2q-1}(U, \mathbb{C})$ and f is absolute q-analytic on $U \setminus f^{-1}(0)$, then Theorem 2.19 implies that $f|_{\lambda}$ is a q-analytic function on $\lambda \cap U$, so invoking Proposition 2.6 we obtain that f is absolute q-analytic on U. This completes the proof of Proposition 2.8. $\square$

A consequence of the characterization given by Proposition 2.6 is a uniqueness property. In finite dimension, it is known that vanishing of infinite order at a single point for a holomorphic function implies vanishing identically. We can generalize this to absolute q-analytic functions.

Corollary 2.10 (To Proposition 2.6). *Let X be a complex Banach manifold, $0 \in X$, and let F be an absolute q-analytic function on X. If there is an open $0 \in U$ satisfying that for every $k \in \mathbb{N}$ there exists a constant $C_k > 0$ such that,*

$$|F(z)| \leq C_k \|z\|^k, \quad z \in U, \tag{16}$$

then $F \equiv 0$.

Proof. Let λ be a one-dimensional complex slice in U passing the origin. By Proposition 2.6, the restriction $f := F|_\lambda$ is q-analytic (in one variable, which we shall denote ζ) near the origin, in particular $f(\zeta) = \sum_{i=0}^{q-1} a_i(\zeta)\bar{\zeta}^i$, where each $a_i(\zeta)$ is holomorphic. Furthermore, Equation (16) implies that for every $k \in \mathbb{N}$ there exists C_k such that $|f(\zeta)| \leq C_k |\zeta|^k$, $z \in U \cap \lambda$. It is sufficient to show that this implies vanishing of f on $\lambda \cap U$. We can use induction in q. If $q = 1$ vanishing of f on $\lambda \cap U$ is immediate due to the well-known property of analytic functions. Let $q > 1$ and assume (as our induction hypothesis) that any $(q-1)$-analytic function $g(\zeta)$ such that for every $k \in \mathbb{N}$ there exists C_k such that $|g(\zeta)| \leq C_k |\zeta|^k$, $z \in U \cap \lambda$, must reduce to the zero function.

Expanding the $a_i(\zeta)$ near the origin we can write with holomorphic a_{il},

$$f(\zeta, \bar{\zeta}) = \sum_{i=1}^{q-1} \sum_{l=0}^{\infty} a_{i,l}(0) \zeta^l \bar{\zeta}^i, \tag{17}$$

for some open ball $D(0, R_0) \subset \mathbb{C}$, $R_0 > 0$. Also,

$$f(\zeta, \bar{\zeta}) = \sum_{i,l} \left(\frac{\partial^{i+l} f(0)}{\partial \bar{\zeta}^j \partial \zeta^l} \right) \zeta^l \bar{\zeta}^i, \quad \zeta \in D(0, R_0), \tag{18}$$

together with $|f(\zeta)| / |\zeta|^k \leq C_k < \infty$, implies that

$$\frac{\partial^{l_1 + l_2} f(0)}{\partial \bar{\zeta}^{l_1} \partial \zeta^{l_2}} = 0, \quad l_1 + l_2 < k, \tag{19}$$

hence

$$\frac{\partial^{l_2} a_{q-1}(0)}{\partial \zeta^{l_2}} = \frac{\partial^{(q-1)+l_2} f(0)}{\partial \bar{\zeta}^{q-1} \partial \zeta^{l_2}}, \tag{20}$$

yields that $a_{q-1}(\zeta)$ vanishes to infinite order at the origin and therefore $a_{q-1} \equiv 0$ (since a_{q-1} is holomorphic). This means that, $f(\zeta) = \sum_{i=0}^{q-2} a_i(\zeta) \bar{\zeta}^i$ thus by the induction hypothesis $f \equiv 0$ on $U \cap \lambda$. This completes the proof. $\square$

2.2. Local uniform limits

Definition 2.11. Let X be a complex Banach space with canonical vector space topology, and let $\mathfrak{M} \subset C(X, \mathbb{C})$ be a family of functions. $\mathfrak{M}$ is said to have the *one dimensional boundary maximum modulus property* if given $f \in \mathcal{M}$, the restriction of f to any complex one dimensional line obeys the boundary maximum modulus principle (in the sense that on the closure of any bounded domain $|f|$ attains maximum on the boundary).

Example 2.12. For any complex submanifold $V \subset X$, $\mathcal{O}(V) \subset C(V, \mathbb{C})$ clearly has the one dimensional boundary maximum modulus property.

We shall be interested in the following families of functions which in particular includes restrictions of absolute q-analytic functions.

Definition 2.13. Let X be a complex Banach space and let $V \subset X$ be a domain. Denote by $\mathfrak{M}(V)$ the set of countably analytic functions g which obey the one-dimensional boundary maximum modulus property of Definition 2.11 such that $\frac{1}{g} \in \mathfrak{M}(V \setminus g^{-1}(0))$. Let $U \subset X$ be a real submanifold. Denote by $\mathfrak{M}_q(U)$ the set of functions f defined on U such that for every $p \in U$ there exists an open $V_p \subset X$ such that f can uniformly approximated[1] on $U \cap V_p$ by absolute q-analytic functions in $\mathfrak{M}(V_p)$.[2]

Example 2.14. Let X denote a complex Banach manifold and $V \subset X$. $\mathfrak{M}_1(V) = \mathcal{O}(V)$, when $V \subset X$ is open.

Example 2.15. Let $U \subset X$ be a complex Banach submanifold. Then,

$$q_1 \le q_2 \Rightarrow \mathfrak{M}_{q_1}(U) \subseteq \mathfrak{M}_{q_2}(U). \tag{21}$$

Example 2.16. A consequence of the definition of $\overline{\mathcal{P}}({}^m X, \mathbb{C})$ is that for any m-homogeneous polynomial $Q_m \in \mathcal{P}_m({}^m X, \mathbb{C})$, $\overline{Q}_m \in \overline{\mathcal{P}}({}^m X, \mathbb{C})$, here we mean the conjugate function, i.e., $Q_m(z) \cdot \overline{Q}_m(z) = |Q_m(z)|^2$. This can be extended to sums of m-homogeneous polynomials with complex

[1]By this we mean convergence in the topology of uniform convergence on compacts, see Appendix.

[2]Note that the reciprocals of the approximating functions need not be absolute q-analytic, merely countably analytic away from their singularities.

coefficients, thus $|P(z)|^2 \in \mathfrak{M}_q(U)$, for any restriction to $U \subset X$, of a polynomial P in z of order q (where U is a real submanifold).

Let X be a complex Hilbert manifold and $M \subset X$ a submanifold. T_pX itself can be given the structure of a complex Hilbert space (it can be identified with X) namely via the linear map $J_p : T_pX \to T_pX$, i.e., $J_p^2 v = -v, \forall v \in T_pX$. Any vector subspace of T_pX which is closed under the application of J_p can then be identified as a complex vector space (with induced complex structure from X). Let $T_p^{\mathbb{C}}M$ denote the largest vector subspace of T_pM which is invariant under the application of J_p, i.e., the largest vector subspace of T_pM which under the induced complex structure is a complex vector subspace of X. Recall that for a C^1-smooth function f the decomposition into $\mathbb{C}$-linear and $\mathbb{C}$-antilinear parts, $df = \partial f + \overline{\partial} f$ implies that f is holomorphic on an open $U \subset \mathbb{C}^n$ iff df_p is $\mathbb{C}$-linear on $T_p\mathbb{C}^n, \forall p \in U$ (in this case $T_p\mathbb{C}^n$ can canonically be equipped with a complex structure).

Let X be a complex Banach space and $M \subset X$ a subspace both with induced topology and differential structure. T_pX itself can be given the structure of a complex Banach space (it can be identified with X) namely via the linear map $J_p : T_pX \to T_pX$ i.e., $J_p^2 v = -v, \forall v \in T_pX$. Any vector subspace of T_pX which is closed under the application of J_p can then be identified as a complex vector space (with induced complex structure from X). Let H_pM (in some literature this is denoted $T_p^{\mathbb{C}}M$ or $T_p^c M$) denote the largest vector subspace of T_pM which is invariant under the application of J_p, i.e., the largest vector subspace of T_pM which under the induced complex structure is a complex vector subspace of X.

Example 2.17. Let X be a complex Hilbert space with open unit ball and unit sphere denoted by $M \subset X$. The unit sphere M is then a real-analytic submanifold with (real) tangent space $T_pM = \{z \in X : \mathrm{Re}(\langle z, p\rangle) = 0\}$, where $\langle ., .\rangle$ denotes the inner product. The maximal complex linear subspace of X contained in T_pM is $H_pM = T_pM \cap iT_pM = \{z \in X : \langle z, p\rangle = 0\}$.

Kaup [10] (2004) introduced what can be interpreted as solutions to tangential Cauchy-Riemann equations in an infinite dimensional setting, in terms of uniform limits of ambient holomorphic functions.

Definition 2.18. Let X be a complex Banach manifold and $M \subset X$ a smooth submanifold. A function $f : M \to \mathbb{C}$ is said to satisfy the *tangential Cauchy-Riemann equations on* M if for all $p \in X$, the differential $df_p : T_pM \to \mathbb{C}$ is complex linear on the subspace $H_pM \subseteq T_pM$. A continuous function $M \to \mathbb{C}$ is to satisfy the tangential Cauchy-Riemann equations on M if it is locally the uniform limit of a sequence of smooth functions that satisfy the tangential Cauchy-Riemann equations on M.

Recently, Daghighi & Wikström [6] introduced a higher finite dimensional more specialized version of the spaces $\mathfrak{M}_q$ which are denoted $\mathfrak{M}_\alpha$, $\alpha \in \mathbb{Z}_+^n$, see Daghighi & Wikström [6].

Definition 2.19. Let $V \subset \mathbb{C}^n$ be a domain. Denote by $\mathfrak{M}(V)$ the set of countably analytic functions g which obey the one dimensional boundary maximum modulus property of Definition 2.11 such that $\frac{1}{g} \in \mathfrak{M}(V \setminus g^{-1}(0))$. Let $U \subset \mathbb{C}^n$ be a real submanifold. Denote by $\mathfrak{M}_\alpha(U)$ the set of functions f defined on U such that for every $p \in U$

there exists an open $V_p \subset \mathbb{C}^n$ such that f can uniformly approximated on $U \cap V_p$ by α-analytic functions in $\mathfrak{M}(V_p)$.[3]

Note that the difference between the possibility of specifying order and absolute order in the one dimensional and the higher (but finite complex dimensional) case was clearly pointed out in Example 2.5. Indeed the q-analytic functions of one variable are precisely the absolute q-analytic function. In light of Proposition 2.6, we have a method of extending result for the spaces $\mathfrak{M}_\alpha(U)$, $\alpha \in \mathbb{Z}_+^n$, where $U \subset \mathbb{C}^n$ to the spaces $\mathfrak{M}_q(U)$, $q \in \mathbb{Z}_+$ and $U \subset X$ for a complex Banach space X. Indeed the restriction of a member of $\mathfrak{M}_q(U)$ to a one dimensional slice λ of U belongs to $\mathfrak{M}_q(\lambda)$.

Definition 2.20. Let X be a complex Banach space and $M \subset X$ be a hypersurface. A point $p_0 \in M$ is called a pseudoconvex point if for any finite two-dimensional complex slice $p_0 \in \mu \subset X$ there is an open $U \subset X$ such that $\mu \cap U$ is pseudoconvex at p_0 in the usual sense.

Clearly, all points of the unit sphere $S \subset X$ of a complex Banach space endowed with the topology induced by the norm, are pseudoconvex points.

Example 2.21. Let X be a complex Banach space endowed with the finite topology and let $M \subset X$ be a hypersurface which is pseudoconvex at $p_0 \in M$. Let further $f \in \mathfrak{M}_q(U)$, for some (relatively) open $p_0 \in U \subset M$. Then there is an open $V \subset X$ and an extension $F \in \mathfrak{M}_q(V \cup U)$, $F|_U = f$. By considering the restrictions to finite dimensional slices passing p_0 it follows from the known result in finite dimension (see Daghighi & Wikström [6]) that if $|f| \equiv$ constant on U, then V contains a nonempty open V' such that $|F| \equiv$ constant on $V' \cup U$.

[3]Note that the reciprocals of the approximating functions need not be α-analytic, merely countably analytic away from their singularities.

Here we must remind the reader that polyanalytic functions of constant modulus on an open set are in general (in contrast to the holomorphic case) nonconstant.

3. Meta-Analytic Functions on Hilbert Manifolds

A generalization of polyanalytic functions of order q are the so called meta-analytic functions.

Definition 3.1 (see, e.g., Balk [2]). Let $\Omega \subset \mathbb{C}$ be a domain and let $S(t) = s_0 + s_1 t + \cdots + s_{n-1} t^{q-1} + t^q$ be a polynomial with complex coefficients. Let $z = x + iy$ denote holomorphic coordinates in $\mathbb{C}^n$. A function $f \in C^q(\Omega, \mathbb{C})$ is called *S-meta-analytic* if it satisfies on Ω the equation $S(\frac{\partial}{\partial \bar{z}})f = 0$.

The following representation is known.

Theorem 3.2 (see Balk [2], p. 239, and references therein). *Let $\Omega \subset \mathbb{C}$ be a domain. If S is a complex polynomial with roots $a_1, \ldots, a_p$ with the multiplicities $m_1, \ldots, m_p$, then a function f is S-meta-analytic in Ω iff $f(z) = \sum_{k=1}^{p} P_k(z) \exp(a_k \cdot \bar{z})$ on Ω, where each P_k is a polyanalytic function* (*with global representation*) *of order m_k.*

For q-analytic functions, there are some known differences regarding the properties of zero sets between the cases $q > 1$ and $q = 1$.

Example 3.3. A set $E \subset \mathbb{C}$ which has a condensation point[4] is not necessarily a set of uniqueness when $q > 1$ (in contrast to the case $q = 1$), see, e.g., Balk [2, p. 207].

[4]Recall that p is a condensation point if for each open neighbourhood U of p the set $U \cap E$ is uncountable.

However, the identity principle remains valid when passing over to meta-analytic functions.

Proposition 3.4. *Let* $\Omega \subset \mathbb{C}^n$ *be a domain. Let* f *and* g *be two* S*-meta-analytic functions on* Ω*. If* $f = g$ *on an open subset* $E \subset \Omega$*, then* $f \equiv g$ *on* Ω*.*

Proof. The easy case of polyanalytic functions in one variable is contained in Example 3.3 (more specifically Balk [2], p. 207). By Theorem 2.7, this immediately yields the result for α-analytic functions (in several variables). Now let $F(z)$ and $G(z)$ be an S_1-meta-analytic functions in one complex variable z (defined on the intersection of the domains of F and G, denoted ω, which w.l.o.g. is assumed to contain the origin) where $S_1(t_1)$ is a polynomial in t_1 with roots $a_1, \dots, a_p$ of associated multiplicities $m_1, \dots, m_p$. By Theorem 3.2, F and G have, on ω, representations

$$F(\zeta) = \sum_{j=1}^{p} P_j(\zeta) \exp(a_j \bar{z}_j), \quad G(\zeta) = \sum_{j=1}^{p} Q_j(\zeta) \exp(a_j \bar{z}_j), \tag{22}$$

where each P_j and Q_j is polyanalytic of order m_j. Now,

$$S_1(\overline{D})(F - G) = 0 \Rightarrow H(z) = (F - G)(z) = \sum_{j=1}^{p} R_j(z) \exp(a_j \bar{z}_j), \tag{23}$$

for some R_j, polyanalytic of order m_j. Assuming $F = G$ on an open subset $E \subset \omega$, we have by Equation (22),

$$\sum_{j=1}^{p} R_j(z) \exp(a_j \bar{z}_j) = 0 \Rightarrow R_p(z) = -\sum_{j=1}^{p-1} R_j(z) \exp(\hat{a}_j \bar{z}_j) \text{ on } E, \tag{24}$$

where $\hat{a}_j = a_j - a_p$, $j = 1, \ldots, p-1$. But $R_p(z)$ is polyanalytic of order m_p so by the identity principle of polyanalytic functions the right hand side must also be polyanalytic of order m_p. However by definition $a_\iota \neq a_\nu$ for $\iota \neq \nu$ thus the openness of E implies that $R_p \equiv 0$ on E, hence $R_p \equiv 0$. So Equation (23) reduces to $(F - G)(z) = \sum_{j=1}^{p-1} R_j(z) \exp(a_j \bar{z}_j)$, and iteration of the arguments become straightforward, yielding $R_j \equiv 0$ for $j = 1, \ldots p$, hence $F - G \equiv 0$. This proves Proposition 3.4. □

A corollary to Theorem 2.9 is the following:

Corollary 3.5. *Let $\Omega \subset \mathbb{C}$ be a domain and let $S(t)$ be a complex polynomial of the form $(a - t)^m$, for a complex constant a and positive integer m. Then any function $f \in C^{2m-1}(U, \mathbb{C})$ which S-meta-analytic on $\Omega \setminus f^{-1}(0)$ is S-meta-analytic on Ω.*

Proof. By Theorem 3.2, we have the representation $f(z) = P(z) \exp (a \cdot \bar{z})$, where each P is polyanalytic of order m on $\Omega \setminus f^{-1}(0)$. In particular $\left(P^{-1}(0)\right) \cap (\Omega \setminus f^{-1}(0)) = \emptyset$. Since f is C^{2m-1}-smooth, $\exp(-a \cdot \bar{z}) f(z)$ is also C^{2m-1}-smooth, in particular, $P(z)$ has C^{2m-1} extension, $\widetilde{P}$, to Ω by defining it to be zero on $f^{-1}(0)$. The function $\widetilde{P}$ satisfies the conditions with respect to Ω of Theorem 2.9 and therefore defines a polyanalytic function of order m on all of Ω. This in turn implies that $P(z) \cdot \exp(a \cdot \bar{z})$ extends to the S-meta-analytic function $\widetilde{P}(z) \cdot \exp(a \cdot \bar{z})$ on Ω. This completes the proof. □

We shall now define an analogue of meta-analytic functions on complex Hilbert manifolds. Let X be a complex Hilbert manifold with inner product denoted $\langle \cdot, \cdot \rangle$ (in particular $\langle v, \zeta z \rangle = \overline{\langle \zeta z, v \rangle} = \bar{\zeta} \langle v, z \rangle$, $v, z \in X$,

$\zeta \in \mathbb{C}$). For each fix $v, w \in X$, we have an anti-linear functional $\langle v, \cdot \rangle : X \to \mathbb{C}$, whose restriction to the complex line $\lambda = \{\zeta w : \zeta \in \mathbb{C}\} \subset X$, defines a bianalytic function $\zeta \mapsto \overline{\zeta}\langle v, w \rangle$.

Definition 3.6. Let X be a complex Hilbert manifold with inner product denoted $\langle \cdot, \cdot \rangle$. Let $A = \{a_1, \dots, a_n\}$ be a set of points in X and let $m = (m_1, \dots, m_n) \in \mathbb{Z}_+^n$. A function $f : X \to \mathbb{C}$ is called (A, m)-meta-analytic at $p_0 \in X$ if there exists an open neighbourhood U of p_0 in X on which f has the representation,

$$f(z) = \sum_{j=1}^{n} f_j(z) \exp\langle z, a_j \rangle, \tag{25}$$

where $f_j(z)$ is an absolute j-analytic function on U, $j = 1, \dots, n$.

A corollary to the fact that absolute q-analytic functions are q-analytic along each one-dimensional complex slice is the following.

Corollary 3.7 (To Proposition 2.6). *Let X be a complex Hilbert manifold. Let $A = \{a_1, \dots, a_k\}$ be a set of points in X and let $m = (m_1, \dots, m_k) \in \mathbb{Z}_+^k$ and let f be a function which is (A, m)-meta-analytic at $p_0 \in X$. Then f is an S_v-meta-analytic function along every one dimensional complex slice $Span_{\mathbb{C}} v$, where S_v is a complex polynomial in one variable, of degree $\sum_j m_j$, such that $\langle v, a_j \rangle \in \mathbb{C}$ is a root of S_v with multiplicity m_j.*

The following is an immediate consequence of Corollary 3.5 together with Corollary 3.7.

Corollary 3.8. *Let X be a complex Hilbert manifold and let* $U \subset X$ *be a bounded domain. Let* $a \in X$, $m \in \mathbb{Z}_+$ *and let* $f \in C^{2m-1}(U)$ *be a function which is* (a, m)*-meta-analytic on* $U \setminus f^{-1}(0)$. *Then f is automatically* (a, m)*-meta-analytic on U.*

Proposition 3.4, immediately yields the infinite dimensional version.

Corollary 3.9. *Let X be a complex Hilbert manifold and let f and g be two S-metaanalytic functions on a domain* $U \subset X$. *If* $f = g$ *on an open subset* $E \subset U$, *then* $f \equiv g$ *on U.*

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Appendix A: Preliminaries on Infinite Dimensional Holomorphy

For the theory of holomorphy in Banach spaces, as it will be presented in this text, please see Dineen [8] and Mujica [15]. Let X be a complex Banach space. By a *domain* $\Omega \subset X$ we mean an open connected set.

Definition A.1 (Continuously differentiable mapping)**.** Let X, Y be two locally convex spaces (e.g., Banach spaces), $\Omega \subset X, \Omega$ open, and $f : \Omega \to Y.$ We say that $f \in C^1($or $f \in C^1(\Omega, Y))$ if

$$df(x, v) = \lim_{\mathbb{R} \ni t \to 0} \frac{f(x + tv) - f(x)}{t},$$

exists for all $(x, v) \in \Omega \times X$ (in particular df is $\mathbb{R}$-linear).

We denote by $cs(X)$ the set of all continuous semi-norms on a topological vector space X. Let X and Y be locally convex vector spaces. Denote by $\mathscr{L}({}^m X, Y)$ the space of m-linear mappings from X^m (the product space) to Y, and we denote by $\mathscr{L}_s({}^m X, Y)$ the vector space of all mappings in $\mathscr{L}({}^m X, Y)$ which are symmetric. To every $\phi \in \mathscr{L}({}^m X, Y)$ (where we do not assume continuity, thus when Y is a scalar field, this is a subset of the algebraic dual) we associate a mapping $\hat{\phi}$ defined by $\hat{\phi} := \phi \cdot x^m$, and call $\hat{\phi}$ the m-homogeneous polynomial associated to ϕ. Denote by $\mathcal{P}({}^m X, Y)$ the sub-vector space of continuous m-homogeneous polynomials. Then the linear mapping from the subspace of continuous functions $\phi \in \mathscr{L}({}^m X, Y)$ to $\mathcal{P}({}^m X, Y)$, defined by $\phi \mapsto \hat{\phi}$, is surjective. Furthermore, the linear mapping from the subspace of continuous functions in $\mathscr{L}_s({}^m X, Y)$ to $\mathcal{P}({}^m X, Y)$ defined by $\phi \mapsto \hat{\phi}$, is bijective.

Definition A.2. By a polynomial, we mean a finite sum of elements in $\bigcup_m \mathcal{P}({}^m X, Y)$, and we will be considering mainly $Y = \mathbb{C}$, and the set of ($\mathbb{C}$-valued continuous) polynomials on X is denoted $\mathcal{P}(X)$.

Definition A.3 (see Mujica [15], p. 33)**.** Let $\Omega \subset X$ be open and nonempty, X locally convex. A function $u : X \to Y$ is called *holomorphic* (or Fréchet holomorphic) if $\forall a \in \Omega, \exists$ a neighbourhood $V \subset U$, and a sequence of polynomials $\{A_m\}_{m \in \mathbb{N}}$, $A_m \in \mathcal{P}({}^m U, Y)$ such that,

$$u(x) = \sum_{m=0}^{\infty} A_m(x - a),$$

uniformly for $x \in V$.

The A_m are of course members of $\mathscr{L}_s({}^m X, Y)$ and if Y is Hausdorff they are uniquely determined by u.

A notion of holomorphy on finitely open subsets on complex Banach spaces, is due to R. Gâteaux, and it holds, see Bremermann [4], that Frechét holomorphic functions are necessarily Gâteaux holomorphic and a Gâteaux holomorphic function is Fréchet holomorphic if it is locally bounded. For our purposes it suffices to know, as is pointed out in Dineen [8], that Hartogs' theorem (in finitely many dimensions) gives that separate holomorphy and local boundedness implies holomorphy, thus $f : E \supset U \to F$ is holomorphic iff $\phi \circ f|_{U \cap U_1}$ is holomorphic for every finite dimensional $U_1 \subset U$.

One can generalize the notion of manifolds by replacing the space they are modelled on.

Definition A.4. A manifold, M, modelled over a Banach space E, is a topological Hausdorff space, paracompact, together with an atlas consisting of charts (ϕ_α, U_α), where $\phi_\alpha : U_\alpha \to V_\alpha$, V_α open in E, is a homeomorphism, and such that the transition maps $\phi_\alpha \circ \phi_\beta^{-1}$ are C^∞ maps on $\phi_\beta(U_\beta \cap U_\alpha)$, $\forall \alpha, \beta$. For a complex Banach manifold, we require that the space is equipped with an atlas of biholomorphically related charts onto open subsets of E and for a real-analytic Banach manifold over a real Banach space we require that it have bi-real analytically related charts onto an open subset of the Banach space.

Banach manifolds of smoothness class k are defined by requiring the ϕ_α to be bijections such that, $\phi_\alpha(U_\alpha)$, $\phi_\alpha(U_\alpha \cap U_\beta)$ are open subsets of some Banach space, and further such that the transition maps are k times continuously differentiable.

Definition A.5 (see Lang [12], p. 25)**.** Let M be a C^k Banach manifold, $x \in X$. Let (ϕ, U), (ψ, V) be two charts at x and $v \in \phi(U)$. (ϕ, U, v), (ψ, V, w) are called *equivalent,* if

$$d(\psi \circ \phi^{-1})_{(\phi(x))v} = w,$$

i.e., the derivative of $\psi \circ \phi^{-1}$ at $\phi(x)$ maps v to w. A *tangent vector* is an equivalence class. The set of tangent vectors of M at x is called the tangent space at x.

The charts on M induce vector bundle charts on the tangent bundle TM so in particular M is a submanifold of TM. A map $f : M_1 \to M_2$ induces a map on the tangent spaces according to, $(x, v) \mapsto (f(x), df_x \cdot v)$, (here we are working in local charts ϕ, φ, for M_1, M_2, respectively, near x and $f(x)$). In general, $T_x M$ will only be a topological vector space.

Appendix B: Topologies

Here we give the very basics on topologies mentioned in the text. First of all the finite topology simply means that open set are the ones satisfying that any finite dimensional slice is open.

Definition B.1. Let $\{p_\beta\}_\beta$ be a family of semi-norms (i.e., a scalar valued, subadditive, nonnegative functions) on a vector space X. Then the sets,

$$\{y \in X|_{p_\beta}(x - y) < r\},\ x \in X,$$

generate a topology (i.e., the topology is the smallest topology containing the given sets) on X, which is called the *topology induced by* the family $\{p_\beta\}_\beta$, of semi-norms.

Definition B.2. A *base for a topology* τ is a collection $W \subset \tau$ such that,

$$\forall U \in \tau,\ \forall x \in U \quad \exists V \in W \text{ such that } x \in V \subseteq U. \tag{26}$$

X is called *locally convex* if it has a basis consisting of convex sets.

We have the following result on semi-norm generated topologies.

Theorem B.3. *If X is a vector space with topology induced from a family of semi-norms, then X is a locally convex topological vector space.*

Definition B.4. Let X, Y be locally convex spaces and $U \subset Y$, be open. *The topology on $\mathcal{O}$ of uniform convergence on compact sets*, is the topology generated by the semi-norms,

$$p_{\beta, K}(f) =: \|f\|_{\alpha,\, K} = \sup \alpha(f(x)), \quad f \in \mathcal{O}(X),$$

where K ranges over the compact subsets of U and α ranges over continuous semi-norms on Y. A basis for this topology is given by the following collection of sets:

$$B_{g, K, \epsilon} = \{f \in \mathcal{O}(\Omega) | \sup_{z \in K} |f(z) - g(z)| < \epsilon\}.$$

Appendix C: Some Preliminaries on Complexification

Here we quickly give some facts on the interplay between real and complex vector space structure. Let V be a real Banach space. Then V is in particular a vector space over $\mathbb{R}$, and can be complexified, by which we mean $\mathbb{C} \otimes_{\mathbb{R}} V$. If we first consider $\mathbb{C} \simeq \mathbb{R}^2$ with standard basis $\{e_1, e_2\}$, scalar multiplication is given by $(a + ib)(x \otimes e_1 + y \otimes e_2) := (ax - by) \otimes e_1 + (bx + ay) \otimes e_2$. $v \mapsto v \otimes e_1$ is an injective real linear map $V \to \mathbb{R}^2 \otimes V$, thus V is a real subspace of $\mathbb{C} \otimes V$. One can prove that $\mathbb{C} \otimes V = V \otimes iV$, which is the reason for the notation, $x \otimes e_1 + y \otimes e_2 =: x + iy$. Complexification induces a conjugation $\overline{av} = \overline{a}v = \overline{a} \otimes v$. The norm of V can be extended in a non-unique way, and two natural requirements on the extension are $\|x\|_{\mathbb{C} \otimes V} = \|x\|_V$, $\forall x \in V$, and $\|x - iy\|_{\mathbb{C} \otimes V} = \|x + iy\|_{\mathbb{C} \otimes V}$, $\forall x, y \in V$. Conversely if we start from a complex Banach space, we can decompose it into $V \otimes iV$, via a projection, $\pi : \mathbb{C} \otimes V \to \mathbb{C} \otimes V$, satisfying $\pi(u + v) = \pi(u) + \pi(v)$, $\pi(au) = a\pi(u)$, $\pi^2(u) = u$. If E is a linear space π a projection on E, such that $\{z \in E : \pi z = z\}$, $\{z \in E\pi(z) = 0\}$ are complex linear manifolds then $\{z \in E : \pi(z) = z\} \bigcap \{z \in E : \pi(z) = 0\} = \emptyset$, and every $z \in E$, has a unique representation $z = z_1 + z_2$, $z_1 \in \{z \in E : \pi(z) = z\}$, $z_2 \in \{z \in E : \pi(z) = 0\}$ (see Taylor [17]).